

# The Long Road to Oslo

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This is the story of how I discovered the work of the Norwegian scientist Sophus Lie who studied continuous groups of finite dimension. I cannot claim to have gained a thorough understanding of his work but I can say that I now appreciate the motivation for it and am myself motivated to pursue it.

## 1 Fractional Derivatives and Group Theory

A few weeks ago I was chatting with my friend Brandon at his place when he shared with me a curious problem. He walked over to his makeshift plexi-glass whiteboard and said (and wrote) something like the following:

The other day I was trying to integrate a nested cosine function, something like  $\cos(a \cos(bx))$ , when I realized that the problem really boils down to computing the integral of nested exponential functions:  $\int_{-\infty}^{\infty} e^{ia e^{ibx}} dx$ . When you try to solve by contour integration, you end up with a pole of a fractional order. Now the well-known residue formula involves a derivative the order of the pole, so I wondered if we could define 'fractional derivatives' to solve this integral, and perhaps they could be interesting in their own right.

Consider a power function  $x^n$ . Taking one derivative we obtain  $nx^{n-1}$ . Another derivative gives  $n(n-1)x^{n-2}$ . We easily generalize this to the  $k$ th order  $\frac{n!}{(n-k)!}x^{n-k}$  for integers  $k$  less than or equal to  $n$ . We should be able to use the gamma function to generalize this formula to  $\frac{\Gamma(n+1)}{\Gamma(n-\eta+1)}x^{n-\eta}$  which seems to be a good guess at the  $\eta$ th order derivative of a function even for non-integer  $\eta$ . For instance, two successive 'half-derivatives' of a power results in a full derivative.

We fooled around with this concept for the next hour, taking fractional derivatives of well-known functions via their power-series expansions, obtaining intuitively satisfying results. For example,  $e^x$  is invariant under fractional derivatives,  $e^{kx}$  becomes  $k^\eta e^{kx}$ , and a sine function smoothly transitions into a cosine as  $\eta$  ranges from 0 to 1, as shown in Figure 1. We got a little bit further,

proving a product-rule using a Fourier integral, before searching for others' work on the subject. Eventually I found the book by Podlubny [1], who examines not only the formalism but also the history and application of derivatives of non-integral order. It turns out that conception of fractional derivatives dates back to Leibniz and more remarkably that there are many applications of fractional differential dynamics to physical problems which are less fruitfully modeled by standard methods.

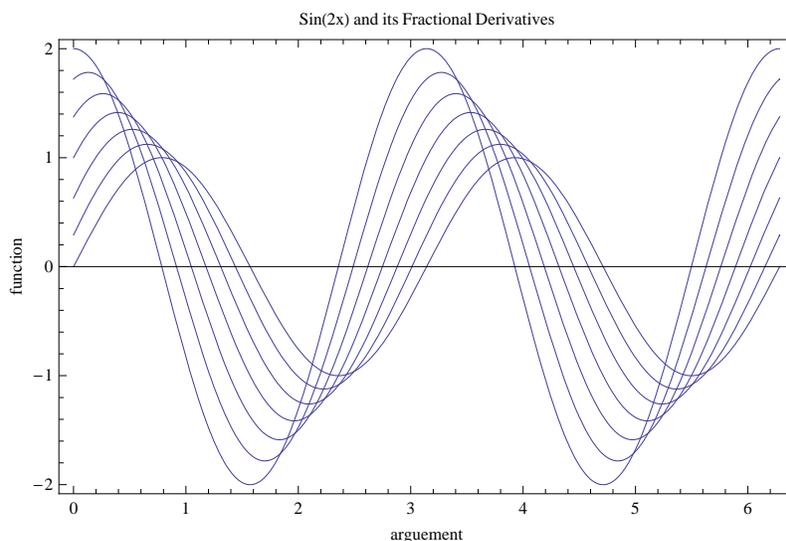


Figure 1: Successive applications of the  $D^{\frac{1}{6}}$  operator to  $\sin(2x)$  gives intuitive results. Note that  $(D^{\frac{1}{6}})^6$  gives the same result as  $D$  in this case, as it should.

Now what does all this have to do with group theory? The connection is this: if we think of the set of derivatives of any order under the composition operation we obtain a binary algebraic structure. Certainly we have closure: composing derivatives gives derivatives of higher order. Furthermore, the identity is a zeroth order derivative. Lastly, the composition of operations is by nature associative and so derivatives form at least a monoid (a group without inverses). However, on a certain class of functions (for example Fourier series with zero bias) derivatives have inverses in the sense that we can take these inverses to be integrals with zero for a constant of integration. The addition of inverses makes derivatives, or perhaps they would be more correctly called *differintegrals*, under composition a group.

We have from calculus a solid definition for differintegrals of integral order. We also know that in general no two differintegrals of different order are equal. Since the composition of two differintegrals is again a differintegral with order the sum of the orders of those entering into the composition (in symbols,  $D^a D^b = D^{a+b}$ ), our integral differintegral group is isomorphic to the integers.

How do fractional derivatives (or differintegrals) come in to play? Well, much as rational and real numbers form additive supergroups of the integers, fractional derivatives should form a supergroup of integral differintegrals. In fact, the fractional ones will be, as a group, isomorphic to the rational or real numbers under addition. What this means is that if we understand how it is that the rationals and/or real numbers may be constructed out of the integers, we should be able to obtain fractional derivatives by analogy.

At this point I became more interested in the general problem of taking a group with ‘macroscopic’ operations such as integral order differintegrals and finding supergroups containing all the fractional operations of the original group. Put in other words, the original group is macroscopic or coarse in the sense that we cannot finely vary from one operation to another as we should be able to with the fractional construction. So the question I set out to address was: can such fine groups be constructed and if so, how?

## 2 Fine Groups and the Construction thereof

Let’s begin with a definition for the sake of clarity. A group  $G$  is said to be *fine* if and only if for each element  $a$  in  $G$  and each rational number  $n/m$  there exists an element  $b$  also in  $G$  satisfying  $a^n = b^m$ . A group which is not fine is said to be *course*.

For example, the second order additive cyclic group  $\mathbb{Z}_3$  (or, to use a Schönflies symbol,  $C_3$ ) is course in particular because there is no ‘cube-root’ for operations other than the identity. However, the circle group (or the group of proper rotations in two dimensions  $C_\infty$ ) contains roots for (i.e. fractional powers of) each operation: if we treat elements of the circle group as phases  $e^{i\theta}$  under multiplication then an  $\frac{n}{m}$ th power of such an element is  $e^{i\frac{n\theta}{m}}$ . Note that a group containing a fine subgroup is not necessarily fine itself, a fact which  $C_{\infty v}$  illustrates nicely.

Fine groups are not necessarily continuous however. For example, complex numbers of unit modulus with rational phases form a fine subgroup of the circle group in an analogous way to how the rationals form a fine subgroup of the reals under addition. This is not in my opinion a serious problem however, because we should be able to apply the method of Dedekind cuts<sup>1</sup> to extend from a fine group to a continuous version (which is itself invariant to such a procedure). Yet, problems of continuity are a rather technical mathematical business and so I will for the most part ignore them here.

Now I turn to what has proved for me so far the most challenging aspect of fine groups: their construction. That is, given a course group, how is it that a fine or continuous group may be constructed so that the course group is a supergroup<sup>2</sup> of the fine construction? Furthermore, how can we demand the

<sup>1</sup>Dedekind’s original description of this method may be found in [2].

<sup>2</sup>Such a request is technically pathological, instead we should ask for a group which contains a subset which is isomorphic to our original group. This is, incidentally, the reason why the term ‘supergroup’ is so seldom used compared to its counterpart, ‘subgroup’.

minimality and uniqueness of the construction? Before we delve into further definitions and attempt a construction, let's examine the examples given in Figure 2.

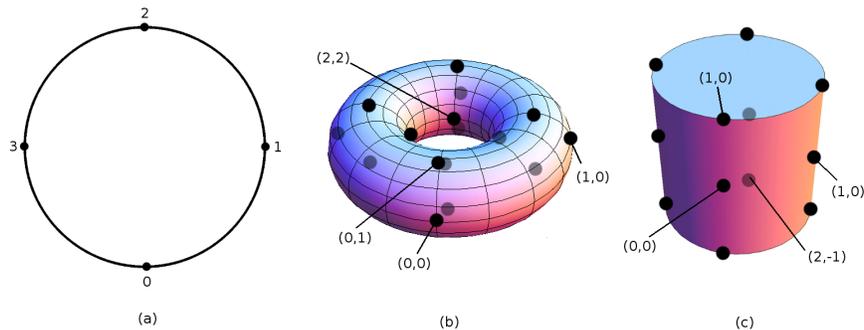


Figure 2: Visualizations of some course groups drawn on those of their proposed minimal fine (continuous) supergroups. The course groups depicted are (a)  $\mathbb{Z}_4$ , (b)  $\mathbb{Z}_4^2$ , (c)  $\mathbb{Z}_4 \times \mathbb{Z}$ . The dots correspond to elements of the course groups. Only some of the dots are labeled to avoid cluttering the figure. Translucent dots indicate they are behind the smooth surface.

After sketching the above figure I set out to find a formal method of construction. For finitely generated abelian groups there is a fairly obvious procedure: with each generator of a minimal generating set associate a dimension of space which is flat or rolled-up on itself if the generator is of infinite or finite order respectively. Such a method suffices to construct the supergroups in Figure 2 but fails when we try to describe non-abelian groups such as those which may generalize to the full 3D rotation group.

Unfortunately, I have so far been unable to find a satisfactory construction to apply to general course groups. A naive attempt to define an equivalence relation on  $G \times \mathbb{Q}$  where  $G$  is a course group fails because of non-uniqueness of roots. Another approach called the field of quotients construction, the canonical algebraic procedure by which the integers are extended to the rationals (see [3] pg 110), will come short of making fine groups when the course group possesses elements of finite order<sup>3</sup>.

But not all is lost for want of a general construction! There is an important sub-problem which is more easily solved, that being the case of finite rotations and translations which form the so-called space groups, taken up in the next section.

<sup>3</sup>The field of quotients technique should be successful in the case of fractional differintegrals, but I have not yet worked this out.

### 3 A Construction for Continuous Space Groups

While for groups in general the construction of fine supergroups is difficult, for a large class of groups useful to physics the construction is straight-forward. Consider a point, line, or space group  $\mathcal{G}$  with the property that each element  $G_i$  may be represented by the combination of an optional inversion, a rotation of  $\theta$  radians about some axis  $\hat{\theta}$ , and a translation of  $x$  units along the  $\hat{x}$  direction. The  $i$ th element  $G_i$  can therefore be represented by the product

$$(-1)^{k_i} R[\vec{\theta}_i] T[\vec{x}_i],$$

where  $\vec{x}$  and  $\vec{\theta}$  denote  $x\hat{x}$  and  $\theta\hat{\theta}$  respectively and  $k_i$  takes on the value 0 or 1.

We now construct a group  $\mathcal{H}$  to be the minimal continuous supergroup of  $\mathcal{G}$  by the following procedure: for each element  $G_i$  from  $\mathcal{G}$  demand that it be in  $\mathcal{H}$  also, along with  $G_i^\alpha$  for all real  $\alpha$ , defined<sup>4</sup> as

$$G_i^\alpha = (-1)^{k_i} R[\alpha\vec{\theta}_i] T[\alpha\vec{x}_i].$$

Then, add to  $\mathcal{H}$  the product elements  $G_i^\alpha G_j^\beta$  so that  $\mathcal{H}$  is closed. Thus  $\mathcal{H}$  forms a minimal supergroup of  $\mathcal{G}$  with continuously variable elements.

An interesting property of  $\mathcal{H}$  is that infinitesimal elements commute. In fact, given some  $H$  in  $\mathcal{H}$ , the space of all elements  $H'$  which differ only infinitesimally from  $H$  is commutative. We call this the *tangent space* to  $\mathcal{H}$  at  $H$ ; and it is Euclidian of a finite dimension so we can find an orthonormal basis for it.

This, however, is as far as I managed to get in my thinking about such groups before reading the history *Symmetry and the Monster* [4] which brought to my attention the work of Sophus Lie and the Lie groups which bear his name.

### 4 Lie Groups

Lie groups are essentially continuous groups of finite dimension, much like those I have been attempting to construct. The study of such groups is a significant subfield of modern algebra; there are many texts on the subject ranging from introductory to the very formal to application-oriented. However, so far as I can tell, any discussion of Lie groups requires the introduction of topology, which is itself a deep and technical field, demanding of intensive study. For instance, the standard definition of Lie group is ‘a topological group that is a differential manifold’ subject to certain smoothness conditions [5]. Notwithstandingly I find this topic intriguing and wish to learn more about it, especially as I have several times heard theoretical physicists mention Lie groups or Lie algebras as important building blocks of contemporary physical theories.

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<sup>4</sup>An alternative to this definition is to compute the root of the representative matrix in general using one of the well-developed matrix rooting procedures. We avoid this here by assuming a form for these matrices.

## 5 Comment on Continuous Composition

Before I close I would like to point out an intriguing possibility afforded by continuous groups which I call *continuous composition*. This phenomenon is most easily demonstrated with the free group<sup>5</sup> on two generators, call them  $a$  and  $b$ . Any element of this group, which I will call  $F$  may be written as an infinite product of the generators  $\prod$

## References

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From MathWorld—A Wolfram Web Resource.  
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<sup>5</sup>freegroup