Extending the Knuth Operator

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Define the **Knuth operator** ($\square$) as a trinary operator over non-negative integers by the following recurrence relation:

$$x \,\square^n\, y = \begin{cases} x + y & \text{if } n = 0 \\ \underbrace{x \,\square^{(n-1)}\, (x \,\square^{(n-1)}\, (x \,...\, \square^{(n-1)}\, x)...))}_{\text{`}x\text{' appears } y \text{ times}} & \text{otherwise} \end{cases}, \tag{1}$$

where x, y, and n are referred to as the **base**, **power**, and **order** of the operator, respectively.

It is easy to see that the zeroth-, first-, and second-order Knuth operators correspond to addition, multiplication, and exponentiation. The third-order Knuth operator can be interpreted as saying 'raise x to the x power y times'. Here is an illustration:

$$2 \,\square^0\, 3 = 2 + 3$$
$$2 \,\square^1\, 3 = 2 \,\square^0\, (2 \,\square^0\, 2) = 2 + 2 + 2 = 2 * 3$$
$$2 \,\square^2\, 3 = 2 \,\square^1\, (2 \,\square^1\, 2) = 2 * 2 * 2 = 2^3$$
$$2 \,\square^3\, 3 = 2 \,\square^2\, (2 \,\square^2\, 2) = 2^{2^2}$$

I would like to see the Knuth operator extended to work over complex numbers. It would be nice to at least generalize to reals, or even just positive rationals. The task of extending the power may be easier than extending the order, but the order extension is more interesting. For example, an extended Knuth operator would elegantly answer the question "What lies between $2 + 3$ and $2 * 3$?" with the evaluation of the expression $2 \,\square^{\frac{1}{2}}\, 3$.

Anyone care to try their hand at this?